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A STOCHASTIC OPTIMAL SEMI-ACTIVE CONTROL STRATEGY FOR ER/MR DAMPERS

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A stochastic optimal semi-active control strategy for randomly excited systems using electrorheological/magnetorheological (ER/MR) dampers is proposed. A system excited by random loading and controlled by using ER/MR dampers is modelled as a controlled, stochastically excited and dissipated Hamiltonian system with n degrees of freedom. The control forces produced by ER/MR dampers are split into a passive part and an active part. The passive control force is further split into a conservative part and a dissipative part, which are combined with the conservative force and dissipative force of the uncontrolled system, respectively, to form a new Hamiltonian and an overall passive dissipative force. The stochastic averaging method for quasi-Hamiltonian systems is applied to the modified system to obtain partially completed averaged Itô stochastic differential equations. Then, the stochastic dynamical programming principle is applied to the partially averaged Itô equations to establish a dynamical programming equation. The optimal control law is obtained from minimizing the dynamical programming equation subject to the constraints of ER/MR damping forces, and the fully completed averaged Itô equations are obtained from the partially completed averaged Itô equations by replacing the control forces with the optimal control forces and by averaging the terms involving the control forces. Finally, the response of semi-actively controlled system is obtained from solving the final dynamical programming equation and the Fokker–Planck–Kolmogorov equation associated with the fully completed averaged Itô equations of the system. Two examples are given to illustrate the application and effectiveness of the proposed stochastic optimal semi-active control strategy.

1. INTRODUCTION

Electrorheological (ER) and magnetorheological (MR) dampers are fluid dampers, the damping forces of which can be controlled by adjusting external electric and magnetic fields, respectively. ER and MR dampers have the capacity to provide large controllable damping forces and several other attractive features such as simplicity, reliability and small power requirement, etc. In the past several years, intensively theoretical and experimental researches on the dynamic behavior and potential application in seismic protection of structures have been made [1, 2]. ER and MR dampers are a class of semi-active control devices. The semi-active control system is a system that utilizes the motion

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of a structure to develop the control force, the magnitude of which can be adjusted by external small power source. Theoretical and experimental researches demonstrated that the performance of a semi-active control system is highly dependent on the choice of control strategy. One challenge in the use of semi-active control technology is to develop non-linear control algorithm that is appropriate for implementation in full-scale structures. A number of semi-active control strategies have been developed [3–8] and several of them have been compared with one another for MR dampers [2].

ER and MR dampers are non-linear semi-active control devices and the dynamic loading experienced by structures is usually random. Thus, non-linear stochastic control algorithm would be appealing. In the past few years, a non-linear stochastic optimal active control strategy has been developed by the present authors [9–12] based on the stochastic averaging method for quasi-Hamiltonian systems [13–15] and the stochastic dynamical programming principle [16–18]. The strategy has several advantages over LQG control and has been extended to partially observable linear structural control [19] and feedback minimization of first-passage failure [20].

In the present paper, the non-linear stochastic optimal active control strategy previously proposed is adapted for its implementation by using ER and MR dampers. A semi-active control system using ER/MR dampers is formulated as a controlled, stochastically excited and dissipated Hamiltonian system. First, the stochastic averaging method for quasi-Hamiltonian systems is applied to the system to obtain the partially completed averaged Itô equations. Then, the stochastic dynamical programming principle is applied to the partially completed averaged system to establish a dynamical programming equation. The optimal control law is obtained from the dynamical programming equation incorporated with the characteristics of ER/MR dampers. Finally, the response of the semi-actively controlled system is obtained by solving the final dynamical programming equation and the Fokker–Planck–Kolmogorov (FPK) equation associated with the fully completed averaged Itô equations. Two examples are given to illustrate the application and effectiveness of the proposed stochastic optimal semi-active control strategy.

2. EQUATIONS OF CONTROLLED SYSTEMS

A system excited by random loading and controlled by using ER/MR dampers is modelled as a controlled, stochastically excited and dissipated Hamiltonian system with n degrees of freedom (d.o.f.). The equations of motion of the system are of the form

$$\begin{aligned} \dot{Q}_i &= \frac{\partial H'}{\partial P_i}, \\ \dot{P}_i &= -\frac{\partial H'}{\partial Q_i} - c'_{ij}\frac{\partial H'}{\partial P_j} + b_{ir}u_r + f_{ik}\xi_k(t), \\ i,j &= 1,2,\ldots,n, \quad r = 1,2,\ldots,s, \quad k = 1,2,\ldots,m, \end{aligned} \tag{1}$$

where Q_i and P_i are the generalized displacements and momenta, respectively, $H' = H'(\mathbf{Q},\mathbf{P})$ is twice differentiable Hamiltonian, $c'_{ij} = c'_{ij}(\mathbf{Q},\mathbf{P})$ represent the coefficients of inherent quasi-linear damping of the system, $f_{ik} = f_{ik}(\mathbf{Q},\mathbf{P})$ represent the magnitudes of random excitations, $\xi_k(t)$ are the random excitations, $u_r = u_r(\mathbf{Q},\mathbf{P})$ represent the control forces produced by ER/MR dampers, b_{ir} are the damper placement coefficients, m and s are the numbers of random excitations and ER/MR dampers respectively.

The control forces u_r produced by ER/MR dampers can be split into passive components u_{rp} and active components u_{ra}, i.e.,

$$u_r(\mathbf{Q},\mathbf{P}) = u_{rp}(\mathbf{Q},\mathbf{P}) + u_{ra}(\mathbf{Q},\mathbf{P}). \tag{2}$$

u_{rp} are the forces produced by ER/MR dampers without external power (zero voltage) while u_{ra} are the increments of the control forces produced by ER/MR dampers due to external power (non-zero voltage).

For example, the dynamic behavior of ER/MR dampers can be quite well described by using the Bingham model [21–23]. According to this model, the control forces produced by ER/MR dampers are

$$u_r = -c_r\dot{X}_r - F_r\,\mathrm{sgn}(\dot{X}_r), \tag{3}$$

where $\dot{X}_r$ are the relative velocities of the two ends of ER/MR dampers; c_r are viscous damping coefficients. $-c_r\dot{X}_r$ are the damping forces of the dampers with zero voltage and thus they are passive control force components. $-F_r\,\mathrm{sgn}(\dot{X}_r)$ are the increments in damping forces due to external voltage and thus they are active control force components. For system (1), one end of each ER or MR damper is assumed to be fixed, independent of the motion of the system. So, $\dot{X}_r$ is the velocity of the system at the location where the rth ER/MR damper is mounted, i.e., $\dot{X}_r = b_{ir}\dot{Q}_i$. Therefore, equation (3) can be rewritten as

$$u_r(\mathbf{Q},\mathbf{P}) = -c_r b_{ir}\dot{Q}_i - F_r\,\mathrm{sgn}(b_{ir}\dot{Q}_i), \tag{4}$$

where there is no summation over r in the first term. F_r are functions of external voltage V_{re}. A simple relationship between F_r and V_{re} proposed by Gavin *et al.* [21] is

$$F_r = C_{ra}V_{re}^{\alpha_r}, \tag{5}$$

where C_{ra} and α_r are the positive constants. Note that an ER or MR damper can only produce the damping force in the opposite direction of the velocity and thus $F_r \geqslant 0$.

In more general case, u_{rp} can further be split into a conservative part and a dissipative part and combined with $-\partial H'/\partial Q_i$ and $-c'_{ij}\partial H'/\partial P_j$ respectively. Then, equation (1) can be rewritten as

$$\begin{aligned}
\dot{Q}_i &= \frac{\partial H''}{\partial P_i},\\
\dot{P}_i &= -\frac{\partial H''}{\partial Q_i} - c''_{ij}\frac{\partial H''}{\partial P_j} + b_{ir}u_{ra} + f_{ik}\xi_k(t),\\
i,j &= 1,2,\ldots,n, \quad r = 1,2,\ldots,s, \quad k = 1,2,\ldots,m,
\end{aligned} \tag{6}$$

where $H'' = H''(\mathbf{Q},\mathbf{P})$ and $c''_{ij} = c''_{ij}(\mathbf{Q},\mathbf{P})$ are the Hamiltonian and damping coefficients modified by the conservative part and dissipative part of the passive control forces of ER/MR dampers respectively.

If $\xi_k(t)$ are Gaussian white noises in the sense of Stratonovich with intensities $2D_{kl}$, equation (6) can be modelled as Stratonovich stochastic differential equations. It can be converted into Itô stochastic differential equations by adding the Wong–Zakai correction terms $D_{kl}f_{jl}\partial f_{ik}/\partial P_j$. These terms can also be split into a conservative part and a dissipative part and then combined with $-\partial H''/\partial Q_i$ and $-c''_{ij}\partial H''/\partial P_j$, respectively, to form overall conservative forces $-\partial H/\partial Q_i$ with a modified Hamiltonian H and $\partial H/\partial P_i = \partial H''/\partial P_i$ and overall dissipative forces $-c_{ij}\partial H/\partial P_j$. With these accomplished, equation (6) becomes

$$
\begin{aligned}
&\mathrm{d}Q_i = \frac{\partial H}{\partial P_i}\mathrm{d}t,\\
&\mathrm{d}P_i = -\left(\frac{\partial H}{\partial Q_i} + c_{ij}\frac{\partial H}{\partial P_j} - b_{ir}u_{ra}\right)\mathrm{d}t + \bar{\sigma}_{ik}\,\mathrm{dB}_k(t),\\
&i,j = 1,2,\ldots,n, \quad r = 1,2,\ldots,s, \quad k = 1,2,\ldots,m,
\end{aligned} \tag{7}
$$

where $\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^{\mathrm{T}} = 2\mathbf{fDf}^{\mathrm{T}}$ and $\mathrm{B}_k(t)$ are independent unit Wiener processes.

3. PARTIALLY AVERAGED SYSTEMS

The n-d.o.f. Hamiltonian system with Hamiltonian H associated with controlled system (7) can be non-integrable, integrable or partially integrable. In the integrable and partially integrable cases, the Hamiltonian system can be further identified as non-resonant or resonant. The stochastic averaging method for quasi-Hamiltonian systems has been developed for all these five cases [13–15]. Due to the limitation of length, here only non-integrable and non-resonant integrable cases are considered. Applying the stochastic averaging method for quasi-Hamiltonian systems to equation (7) we obtain

$$
\mathrm{d}H = \left[m(H) + \left\langle \frac{\partial H}{\partial P_i} b_{ir}u_{ra}\right\rangle\right]\mathrm{d}t + \sigma(H)\,\mathrm{d}B(t), \tag{8}
$$

where

$$
\begin{aligned}
&m(H) = \frac{1}{T(H)V_{\Omega_1}}\int_{\Omega}\left[\left(-c_{ij}\frac{\partial H\partial H}{\partial p_i\,\partial p_j} + \frac{1}{2}\bar{\sigma}_{ik}\bar{\sigma}_{jk}\frac{\partial^2 H}{\partial p_i\partial p_j}\right)\Big/\frac{\partial H}{\partial p_1}\right]\mathrm{d}q_1\cdots\mathrm{d}q_n\,\mathrm{d}p_2\cdots\mathrm{d}p_n,\\
&\sigma^2(H) = \frac{1}{T(H)V_{\Omega_1}}\int_{\Omega}\left(\bar{\sigma}_{ik}\bar{\sigma}_{jk}\frac{\partial H\partial H}{\partial p_i\,\partial p_j}\Big/\frac{\partial H}{\partial p_1}\right)\mathrm{d}q_1\cdots\mathrm{d}q_n\,\mathrm{d}p_2\cdots\mathrm{d}p_n,\\
&T(H) = \frac{1}{V_{\Omega_1}}\int_{\Omega}\left(1\Big/\frac{\partial H}{\partial p_1}\right)\mathrm{d}q_1\cdots\mathrm{d}q_n\,\mathrm{d}p_2\cdots\mathrm{d}p_n,\\
&V_{\Omega_1} = \int_{\Omega_1}\mathrm{d}q_2\cdots\mathrm{d}q_n\,\mathrm{d}p_2\cdots\mathrm{d}p_n,\\
&\Omega = \{(q_1,\ldots,q_n,p_2,\ldots,p_n)|H(q_1,\ldots,q_n,0,p_2,\ldots,p_n)\leqslant H\},\\
&\Omega_1 = \{(q_2,\ldots,q_n,p_2,\ldots,p_n)|H(0,q_2,\ldots,q_n,0,p_2,\ldots,p_n)\leqslant H\}.
\end{aligned} \tag{9}
$$

for non-integrable case, or

$$
\begin{aligned}
\mathrm{d}H_\alpha &= \left[m_\alpha(\mathbf{H}) + \left\langle\frac{\partial H_\alpha}{\partial P_i}b_{ir}u_{ra}\right\rangle\right]\mathrm{d}t + \sigma_{\alpha k}(\mathbf{H})\,\mathrm{dB}_k(t),\\
\alpha &= 1,2,\ldots,n, \quad k = 1,2,\ldots,m,
\end{aligned} \tag{10}
$$

where $\mathbf{H} = [H_1, H_2, \ldots, H_n]$, H_α are first integrals,

$$
\begin{aligned}
&m_\alpha(\mathbf{H}) = \left\langle -c_{ij}\frac{\partial H}{\partial p_j}\frac{\partial H_\alpha}{\partial p_i} + \frac{1}{2}\bar{\sigma}_{ik}\bar{\sigma}_{jk}\frac{\partial^2 H_\alpha}{\partial p_i\partial p_j}\right\rangle_t,\\
&\sigma_{\alpha k}(\mathbf{H})\sigma_{\beta k}(\mathbf{H}) = \left\langle \bar{\sigma}_{ik}\bar{\sigma}_{jk}\frac{\partial H_\alpha}{\partial p_i}\frac{\partial H_\beta}{\partial p_j}\right\rangle_t
\end{aligned} \tag{11}
$$

for non-resonant integrable case. In equations (8) and (10), $\langle\ \rangle$ denote the averaging defined by the first equations of equations (9) and (11) respectively. In equation (11), $\langle\cdot\rangle_t$ denotes the time averaging [14].

4. OPTIMAL CONTROL LAW

Equations (8) and (10) imply that in non-integrable case, Hamiltonian H is a one-dimensional controlled diffusion process and in non-resonant integrable case $\mathbf{H}$ is an n-dimensional vector of controlled diffusion processes respectively. The optimal control law depends on the objective of system control, which is expressed in terms of performance index. Here we consider the response control in finite and in infinite time intervals. In the case of finite time interval control, the performance index is assumed to be of the Boltz type, i.e.,

$$J = E\left[\int_0^{t_f} L(H(\tau), \langle \mathbf{u}_a(\tau)\rangle)\,\mathrm{d}\tau + \Psi(H(t_f))\right] \tag{12}$$

for system (8), or

$$J = E\left[\int_0^{t_f} L(\mathbf{H}(\tau), \langle \mathbf{u}_a(\tau)\rangle)\,\mathrm{d}\tau + \Psi(\mathbf{H}(t_f))\right] \tag{13}$$

for system (10). In equations (12) and (13), $\mathbf{u}_a = [u_{1a}, u_{2a}, \ldots, u_{sa}]^{\mathrm{T}}$, L is the cost function, t_f is the final time of control and $\Psi(t_f)$ is the final cost. In the case of infinite time interval control, the performance index is usually of average type, i.e.,

$$J = \lim_{T\to\infty} \frac{1}{T}\int_0^{T} L(H(\tau), \langle \mathbf{u}_a(\tau)\rangle)\,\mathrm{d}\tau \tag{14}$$

for system (8), or

$$J = \lim_{T\to\infty} \frac{1}{T}\int_0^{T} L(\mathbf{H}(\tau), \langle \mathbf{u}_a(\tau)\rangle)\,\mathrm{d}\tau \tag{15}$$

for system (10). The control of system (8) or (10) with performance index (14) or (15) is called the ergodic control.

Here, some physical interpretations are relevant. For most dynamical systems, H is the total energy of a system. So, cost function $L(H, \mathbf{u}_a)$ is a function of total system energy and the active components of control forces produced by ER/MR dampers. For linear Hamiltonian systems, H is a quadratic function of displacements and velocities. In this case, L is a quadratic function of system state, which is similar to that used in the usual LQR or LQG control. $\mathbf{H}$ is a vector of conservative quantities of the Hamiltonian (conservative) system associated with equation (7). Thus, cost function $L(\mathbf{H}, \mathbf{u}_a)$ is a function of these conservative quantities and the active components of control forces produced by ER/MR dampers. The objective of the stochastic optimal semi-active control is to minimize the performance index J in one of equations (12)–(15) by properly designing the control law for $\mathbf{u}_a$.

Based on the stochastic dynamical programming principle [16–18], a dynamical programming equation can be set up. It is

$$\frac{\partial V}{\partial t} = -\min_{\mathbf{u}_a}\left\{ L(H, \langle \mathbf{u}_a\rangle) + \left[m(H)\left\langle + \frac{\partial H}{\partial p_i} b_{ir} u_{ra}\right\rangle\right]\frac{\partial V}{\partial H} + \frac{\sigma^2(H)}{2}\frac{\partial^2 V}{\partial H^2}\right\} \tag{16}$$

for controlled system (8) with performance index (12), or

$$\frac{\partial V}{\partial t} = -\min_{\mathbf{u}_a}\left\{ L(\mathbf{H}, \langle \mathbf{u}_a\rangle) + \left[m_\alpha(\mathbf{H}) + \left\langle \frac{\partial H_\alpha}{\partial p_i} b_{ir} u_{ra}\right\rangle\right]\frac{\partial V}{\partial H_\alpha} + \frac{1}{2}\sigma_{\alpha k}(\mathbf{H})\sigma_{\beta k}(\mathbf{H})\frac{\partial^2 V}{\partial H_\alpha \partial H_\beta}\right\} \tag{17}$$

for controlled system (10) with performance index (13). In equations (16) and (17), $V = V(H, t)$ and $V(\mathbf{H}, t)$ are called value functions. For ergodic control of systems (8) and (10), the dynamical programming equations are

$$\lambda = \min_{\mathbf{u}_a}\left\{ L(H, \langle \mathbf{u}_a \rangle) + \left[m(H) + \left\langle \frac{\partial H}{\partial p_i} b_{ir} u_{ra} \right\rangle \right] \frac{\partial V}{\partial H} + \frac{\sigma^2(H)}{2}\frac{\partial^2 V}{\partial H^2} \right\} \tag{18}$$

and

$$\lambda = \min_{\mathbf{u}_a}\left\{ L(\mathbf{H}, \langle \mathbf{u}_a \rangle) + \left[m_\alpha(\mathbf{H}) + \left\langle \frac{\partial H_\alpha}{\partial p_i} b_{ir} u_{ra} \right\rangle \right] \frac{\partial V}{\partial H_\alpha} + \frac{1}{2}\sigma_{\alpha k}(\mathbf{H})\sigma_{\beta k}(\mathbf{H}) \frac{\partial^2 V}{\partial H_\alpha \partial H_\beta} \right\} \tag{19}$$

respectively.

The optimal control law for system (8) is obtained from minimizing the right-hand side of equation (16) or (18), i.e.,

$$\frac{\partial}{\partial u_{ra}}\left[L(H, \langle \mathbf{u}_a \rangle) + \left\langle \frac{\partial H}{\partial p_i} b_{ir} u_{ra} \right\rangle \frac{\partial V}{\partial H} \right] = 0. \tag{20}$$

Suppose that the cost function is of the form

$$L(H, \langle \mathbf{u}_a \rangle) = g(H) + \langle \mathbf{u}_a^{\mathrm{T}} \mathbf{R} \mathbf{u}_a \rangle, \tag{21}$$

where $g(H) \geqslant 0$ and $\mathbf{R}$ is a positive-definite symmetric matrix. Then the optimal control law is of the form

$$u_{ra}^* = -\frac{1}{2} R_{rs}^{-1} \frac{\partial V}{\partial H} b_{is} \frac{\partial H}{\partial P_i} = -\frac{1}{2} R_{rs}^{-1} \frac{\partial V}{\partial H} b_{is} \dot{Q}_i, \tag{22}$$

which can be rewritten as

$$u_{ra}^* = -F_{ra}^* \operatorname{sgn}(b_{ir} \dot{Q}_i) \tag{23}$$

with

$$F_{ra}^* = \frac{1}{2} R_{rs}^{-1} b_{is} \dot{Q}_i \operatorname{sgn}(b_{jr} \dot{Q}_j) \frac{\partial V}{\partial H}. \tag{24}$$

Note that u_{ra}^* in equation (22) or equations (23) and (24) are the optimal control forces required by system (8). On the other hand, the active control forces produced by ER/MR dampers are of the form of the second term in equation (4) with positive F_r. It is seen that the necessary optimal control forces u_{ra}^* can be produced by ER/MR dampers with sufficient capacity if $F_{ra}^* \geqslant 0$. In fact, letting F_{ra}^* be equal to F_r in equation (5) yields the regulation law for external voltage V_{re}, i.e., $V_{re} = (F_{ra}^*/C_{ra})^{-\alpha_r}$. However, if $F_{ra}^* < 0$, the desired optimal control forces u_{ra}^* cannot be produced by ER/MR dampers. In this case, the external voltage is set to be zero, i.e., active control is clipped. Obviously, the longer the time for $F_{ra}^* \geqslant 0$ is, the more effective the ER/MR dampers are. For example, suppose that $\mathbf{R}$ is taken to be positive-definite diagonal matrix and $g(H)$ and λ are selected so that $\partial V / \partial H \geqslant 0$. In this case equation (22) becomes

$$u_{ra}^* = -\frac{1}{2R_{rr}} \frac{\partial V}{\partial H} b_{ir} \dot{Q}_i = -\frac{1}{2R_{rr}} \frac{\partial V}{\partial H} |b_{ir} \dot{Q}_i| \operatorname{sgn}(b_{ir} \dot{Q}_i) \tag{25}$$

and the dampers may produce exactly the necessary optimal control forces all the time.

For system (10), the optimal control law can be obtained similarly and equations (20)–(25) are replaced by

$$\frac{\partial}{\partial u_{ra}}\left[L(\mathbf{H}, \langle \mathbf{u}_a \rangle) + \left\langle \frac{\partial H_\alpha}{\partial p_i} b_{ir} u_{ra} \right\rangle \frac{\partial V}{\partial H_\alpha} \right] = 0, \tag{26}$$

$$L(\mathbf{H}, \langle \mathbf{u}_a \rangle) = g(\mathbf{H}) + \langle \mathbf{u}_a^{\mathrm{T}} \mathbf{R} \mathbf{u}_a \rangle, \tag{27}$$

$$u^*_{ra} = -\frac{1}{2} R^{-1}_{rs} b_{is} \frac{\partial V}{\partial H_\alpha} \frac{\partial H_\alpha}{\partial P_i}, \qquad u^*_{ra} = -F^*_{ra} \operatorname{sgn}(b_{ir} \dot{Q}_i), \tag{28, 29}$$

$$F^*_{ra} = \frac{1}{2} R^{-1}_{rs} b_{is} \frac{\partial V}{\partial H_\alpha} \frac{\partial H_\alpha}{\partial P_i} \operatorname{sgn}(b_{jr} \dot{Q}_j), \tag{30}$$

$$u^*_{ra} = -\frac{1}{2R_{rr}} \frac{\partial V}{\partial H_i} b_{ir} \dot{Q}_i = -\frac{1}{2R_{rr}} \frac{\partial V}{\partial H_i} |b_{ir} \dot{Q}_i| \operatorname{sgn}(b_{ir} \dot{Q}_i) \tag{31}$$

respectively.

5. RESPONSE PREDICTION

Substituting the optimal control forces u^*_{ra} into dynamical programming equations (16)–(19) and averaging the terms involving active control forces yield the final dynamical programming equations. Associated with equations (16)–(19), they are

$$\frac{\partial V}{\partial t} + L(H, \langle \mathbf{u}^*_a \rangle) + \left[m(H) + \left\langle \frac{\partial H}{\partial P_i} b_{ir} u^*_{ra} \right\rangle \right] \frac{\partial V}{\partial H} + \frac{\sigma^2(H)}{2} \frac{\partial^2 V}{\partial H^2} = 0, \tag{32}$$

$$\frac{\partial V}{\partial t} + L(\mathbf{H}, \langle \mathbf{u}^*_a \rangle) + \left[m_\alpha(\mathbf{H}) + \left\langle \frac{\partial H_\alpha}{\partial P_i} b_{ir} u^*_{ra} \right\rangle \right] \frac{\partial V}{\partial H_\alpha} + \frac{1}{2} \sigma_{\alpha k}(\mathbf{H}) \sigma_{\beta k}(\mathbf{H}) \frac{\partial^2 V}{\partial H_\alpha \partial H_\beta} = 0, \tag{33}$$

$$\lambda = L(H, \langle \mathbf{u}^*_a \rangle) + \left[m(H) + \left\langle \frac{\partial H}{\partial P_i} b_{ir} u^*_{ra} \right\rangle \right] \frac{\partial V}{\partial H} + \frac{\sigma^2(H)}{2} \frac{\partial^2 V}{\partial H^2}, \tag{34}$$

$$\lambda = L(\mathbf{H}, \langle \mathbf{u}^*_a \rangle) + \left[m_\alpha(\mathbf{H}) + \left\langle \frac{\partial H_\alpha}{\partial P_i} b_{ir} u^*_{ra} \right\rangle \right] \frac{\partial V}{\partial H_\alpha} + \frac{1}{2} \sigma_{\alpha k}(\mathbf{H}) \sigma_{\beta k}(\mathbf{H}) \frac{\partial^2 V}{\partial H_\alpha \partial H_\beta} \tag{35}$$

respectively.

On the other hand, substituting optimal control forces u^*_{ra} into partially averaged Itô equations (8) and (10) and averaging the terms involving active control forces lead to fully averaged Itô equations. Associated with equations (8) and (10), they are

$$\mathrm{d}H = \bar{m}(H)\,\mathrm{d}t + \sigma(H)\,\mathrm{dB}(t), \tag{36}$$

where

$$\bar{m}(H) = m(H) + \left\langle \frac{\partial H}{\partial P_i} b_{ir} u^*_{ra} \right\rangle \tag{37}$$

and

$$\mathrm{d}H_\alpha = \bar{m}_\alpha(\mathbf{H})\,\mathrm{d}t + \sigma_{\alpha k}(\mathbf{H})\,\mathrm{dB}_k(t), \tag{38}$$

where

$$\bar{m}_\alpha(\mathbf{H}) = m_\alpha(\mathbf{H}) + \left\langle \frac{\partial H_\alpha}{\partial P_i} b_{ir} u^*_{ra} \right\rangle_t. \tag{39}$$

Note that $\bar{m}(H)$ and $\bar{m}_\alpha(\mathbf{H})$ involve unknown $\partial V/\partial H$ and $\partial V/\partial H_\alpha$, which are obtained by solving equations (32) or (34), and (33) or (35) respectively.

Solving the FPK equations associated with Itô equations (36) and (38), we obtain the response of semi-actively controlled system. The response of passively controlled system can be obtained from solving the FPK equations without active control forces. The statistics of active control force components can be obtained by the probability density of the response of semi-actively controlled system and equation (22) or (28).

To evaluate a semi-active control strategy, two measures are introduced. One is the percentage reduction in the root-mean-square displacement due to the active control force components produced by ER/MR dampers, i.e., effectiveness of a semi-active control strategy, which is defined as

$$\mathrm{K}_s = \frac{\sqrt{E[X_p^2]} - \sqrt{E[X_s^2]}}{\sqrt{E[X_p^2]}} \times 100\%, \tag{40}$$

where $E[X_s^2]$ and $E[X_p^2]$ denote the mean square displacements of semi-actively and passively controlled systems respectively. The other measure is the efficiency of a semi-active control strategy, which is defined as the ratio of the percentage reduction K_s to the normalized sum of the root-mean-square active control force components produced by ER/MR dampers, i.e.,

$$\mu_s = \frac{\mathrm{K}_s}{\sum_{r=1}^{s} \sqrt{E[u_{ra}^{*2}]} / \sum_{k=1}^{m} \sqrt{2D_{kk}}}, \tag{41}$$

where $E[u_{ra}^{*2}]$ are the mean square values of active control force components produced by ER/MR dampers. Note that u_{ra}^* in μ_s imply external input voltage V_{re}^* since $u_{ra} = -F_r \operatorname{sgn}(b_{ir}\dot{Q}_i)$ and $F_r = C_{ra} V_{re}^{\alpha_r}$ [see equations (4) and (5)]. The larger the K_s and μ_s are, the better the semi-active control strategy is.

6. EXAMPLE 1

As a simple example, consider a Duffing oscillator subject to Gaussian white-noise excitation and controlled by an ER or MR damper. The equation of motion of the system is

$$\ddot{X} + c_0 \dot{X} + aX + bX^3 = \xi(t) + u, \tag{42}$$

where c_0, a and b are the constants, and $\xi(t)$ is a Gaussian white noise with intensity D. The control force u is produced by an ER or MR damper. The passive control force $-c_1\dot{X}$ of the ER or MR damper can be combined with inherent damping force $-c_0\dot{X}$ to form $-c\dot{X}$. Thus, equation (42) becomes

$$\ddot{X} + c\dot{X} + aX + bX^3 = \xi(t) + u_a, \tag{43}$$

which can be rewritten as

$$\dot{Q} = P, \qquad \dot{P} = -cP - aQ - bQ^3 + u_a + \xi(t). \tag{44}$$

Using the stochastic averaging method of energy envelope [24], the partially completed averaged Itô equation can be obtained as follows:

$$\mathrm{d}H = \left[m(H) + \frac{\partial H}{\partial P} u_a \right] \mathrm{d}t + \sigma(H)\, \mathrm{d}B(t), \tag{45}$$

where

$$H = P^2/2 + aQ^2/2 + bQ^4/4,$$
$$m(H) = D/2 - cG(H), \qquad \sigma^2(H) = DG(H),$$
$$G(H) = \frac{4}{T(H)}\int_0^A (2H - aq^2 - bq^4/2)^{1/2}\,\mathrm{d}q,$$
$$T(H) = 4\int_0^A (2H - aq^2 - bq^4/2)^{-1/2}\,\mathrm{d}q,$$
$$A = \{[(a^2 + 4bH)^{1/2} - a]/b\}^{1/2}. \tag{46}$$

Let the cost function be quadratic in u_a and polynomial in H, i.e.,

$$L = g(H) + R\langle u_a^2\rangle, \tag{47}$$

where R is a positive constant and

$$g(H) = s_0 + s_1 H + s_2 H^2 + s_3 H^3. \tag{48}$$

Following equation (23), the optimal control force is

$$u_a^* = -F_a^* \operatorname{sgn}(\dot{Q}), \quad F_a^* \geqslant 0, \tag{49}$$

where

$$F_a^* = \frac{1}{2R}\frac{\mathrm{d}V}{\mathrm{d}H}\frac{\partial H}{\partial P}\operatorname{sgn}(\dot{Q}) = \frac{1}{2R}\frac{\mathrm{d}V}{\mathrm{d}H}|\dot{Q}|. \tag{50}$$

For ergodic control, $\mathrm{d}V/\mathrm{d}H$ is obtained from solving the following final dynamical programming equation:

$$\lambda = g(H) + m(H)\frac{\mathrm{d}V}{\mathrm{d}H} - \frac{1}{4R}G(H)\left(\frac{\mathrm{d}V}{\mathrm{d}H}\right)^2 + \frac{1}{2}\sigma^2(H)\frac{\mathrm{d}^2V}{\mathrm{d}H^2}. \tag{51}$$

There exists a relationship $s_0 = \lambda - (D/2)\,\mathrm{d}V/\mathrm{d}H$ when $H = 0$. It is always possible to select s_0, s_1, s_2, s_3, R and λ so that $\mathrm{d}V/\mathrm{d}H \geqslant 0$. In this case, the damper produces the necessary optimal control force all the time only if the damper has sufficient capacity.

Solving equation (51) yields $\mathrm{d}V/\mathrm{d}H$ and substituting it into equations (49) and (50) yields u_a^*. Substituting u_a^* into equation (45) to replace u_a and averaging $(\partial H/\partial P)u_a^*$ lead to the fully averaged Itô equation for H. The stationary probability density of total energy of the semi-actively controlled system is then obtained from solving the reduced FPK equation associated with the fully completed averaged Itô equation. It is

$$p_s(H) = C_s \exp\left\{-\int_0^H \left[\left(-2m(y) + \frac{G(y)}{R}\frac{\mathrm{d}V}{\mathrm{d}y} + \frac{\mathrm{d}\sigma^2(y)}{\mathrm{d}y}\right)\Big/\sigma^2(y)\right]\mathrm{d}y\right\}. \tag{52}$$

The stationary probability density of total energy of the passively controlled system is obtained from equation (52) by making the active-control-induced term vanishing, i.e.,

$$p_p(H) = C_p \exp\left\{-\int_0^H \left[\left(-2m(y) + \frac{\mathrm{d}\sigma^2(y)}{\mathrm{d}y}\right)\Big/\sigma^2(y)\right]\mathrm{d}y\right\}. \tag{53}$$

The mean square displacements of semi-actively and passively controlled systems can be obtained from equations (52) and (53), respectively, as follows:

$$E[X_s^2] = C_s\int_0^\infty \frac{p_s(H)}{T(H)}\,\mathrm{d}H\int_0^A \frac{4q^2\,\mathrm{d}q}{(2H - aq^2 - bq^4/2)^{1/2}}, \tag{54}$$

$$E[X_p^2] = C_p \int_0^\infty \frac{p_p(H)}{T(H)} \mathrm{d}H \int_0^A \frac{4q^2\,\mathrm{d}q}{(2H - aq^2 - bq^4/2)^{1/2}}, \tag{55}$$

while the mean square active control force component is obtained from equations (49), (50) and (52) as follows:

$$E[u_a^{*2}] = C_s \int_0^\infty \frac{G(H)}{4R} \left(\frac{\mathrm{d}V}{\mathrm{d}H}\right)^2 p_s(H)\,\mathrm{d}H. \tag{56}$$

To see the merit of the proposed stochastic optimal semi-active control strategy, system (42) is restudied by using the clipped LQG control. For this purpose, equation (43) is first linearized by using the statistical linearization method. The linearized equation is

$$\ddot{X} + c\dot{X} + (a + 3bE[X^2])X = \xi(t) + u_a \tag{57}$$

and it can be rewritten as

$$\mathrm{d}X/\mathrm{d}t = \dot{X}, \qquad \mathrm{d}\dot{X}/\mathrm{d}t = -c\dot{X} - (a + 3bE[X^2])X + u_a + \xi(t). \tag{58}$$

Let the cost function be

$$L = \mathbf{Y}^{\mathrm{T}}\mathbf{S}\mathbf{Y} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}, \tag{59}$$

where

$$\mathbf{Y} = \begin{bmatrix} X \\ \dot{X} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} s_{11} & 0 \\ 0 & s_{22} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ u_a \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}. \tag{60}$$

The dynamical programming equation for ergodic control is of the form

$$\lambda = \min_{u_a} \left\{ L + \dot{X}\frac{\partial V}{\partial X} + [-c\dot{X} - (a + 3bE[X^2])X + u_a]\frac{\partial V}{\partial \dot{X}} + \frac{D}{2}\frac{\partial^2 V}{\partial \dot{X}^2} \right\}. \tag{61}$$

The required optimal control force is obtained from minimizing the right-hand side of equation (61) as follows:

$$u_a^* = -\frac{1}{2R}\frac{\partial V}{\partial \dot{X}}. \tag{62}$$

The solution of equation (61) is of the form

$$V = \mathbf{Y}^{\mathrm{T}}\mathbf{P}\mathbf{Y}, \tag{63}$$

where

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \tag{64}$$

can be obtained from solving the following Riccati equation

$$\mathbf{S} + \mathbf{P}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{P}\mathbf{R}^{-1}\mathbf{P} = 0, \tag{65}$$

in which

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -(a + 3bE[X^2]) & -c \end{bmatrix}, \quad \mathbf{R}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 1/R \end{bmatrix}. \tag{66}$$

Substituting equation (63) into equation (62) yields the necessary optimal control force

$$u_a^* = -\frac{1}{R}(P_{12}X + P_{22}\dot{X}). \tag{67}$$

As noted previously, an ER or MR damper can only produce active control force

$$u_a = -F\,\mathrm{sgn}(\dot{X}). \tag{68}$$

So, the damper produces active control force only when

$$\frac{1}{R}(P_{12}X + P_{22}\dot{X})\,\mathrm{sgn}(\dot{X}) \geqslant 0 \tag{69}$$

If condition (69) is satisfied, then the input external voltage is determined for example, by equation (5). If condition (69) is not satisfied, u_a^* should vanish. In this case, the response of semi-actively controlled system can be calculated numerically. Note that the clipped LQG control strategy here is different from that in reference [2]. In the special case where $s_{11} = 0$, $P_{12} = 0$, equation (69) will be satisfied all the time if $P_{22} > 0$. The semi-actively controlled system is of the form

$$\ddot{X} + (c + \bar{P}_{22}/R)\dot{X} + (a + 3bE[X^2] + \bar{P}_{12}/R)X = \xi(t), \tag{70}$$

where $\bar{P}_{12}$ and $\bar{P}_{22}$ are the coefficients of linearized control force corresponding to P_{12} and P_{22} respectively. Denote

$$\mathbf{W} = \begin{bmatrix} E[X^2] & E[X\dot{X}] \\ E[\dot{X}X] & E[\dot{X}^2] \end{bmatrix}. \tag{71}$$

Covariance matrix $\mathbf{W}$ satisfies the following Lyapunov equation:

$$\bar{\mathbf{A}}\mathbf{W} + \mathbf{W}\bar{\mathbf{A}}^{\mathrm{T}} = -\mathbf{D}, \tag{72}$$

where

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ -(a + 3bE[X^2] + \bar{P}_{12}/R) & -(c + \bar{P}_{22}/R) \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}. \tag{73}$$

The clipped LQG control force u_a^* and the response of semi-actively controlled system can be obtained by solving equations (65) and (72) simultaneously.

The numerical results for the example system (42) have been obtained by using the proposed stochastic optimal semi-active control strategy and the clipped LQG control strategy. In Figures 1–4, $a = 1{\cdot}0$, $c = 0{\cdot}18$; for the proposed control strategy, $R = 1{\cdot}0$, $s_1 = s_3 = 0$, $s_2 = 1{\cdot}0$, $\mathrm{d}V/\mathrm{d}H|_{H=0} = 2{\cdot}5$; for the clipped LQG control strategy, $R = 1{\cdot}0$, $s_{11} = 6{\cdot}0$, $s_{22} = 0$. It is seen from these figures that the proposed stochastic optimal semi-active control strategy is much better than the clipped LQG control strategy in terms of both measures K_s and μ_s.

7. EXAMPLE 2

As a more complicated example, consider a hysteretic column subject to both horizontal and vertical random ground acceleration excitations and controlled by an ER or MR damper. The equation of motion of the system is

$$\ddot{X} + 2\zeta_0\dot{X} + [\alpha - k_1 - k_2\eta(t)]X + (1 - \alpha)Z = \xi(t) + u, \tag{74}$$

where X denotes non-dimensional displacement, ζ_0 is the viscous damping ratio, α is the ratio of stiffness after yield to stiffness before yield, and k_1 and k_2 are the constants. $\xi(t)$

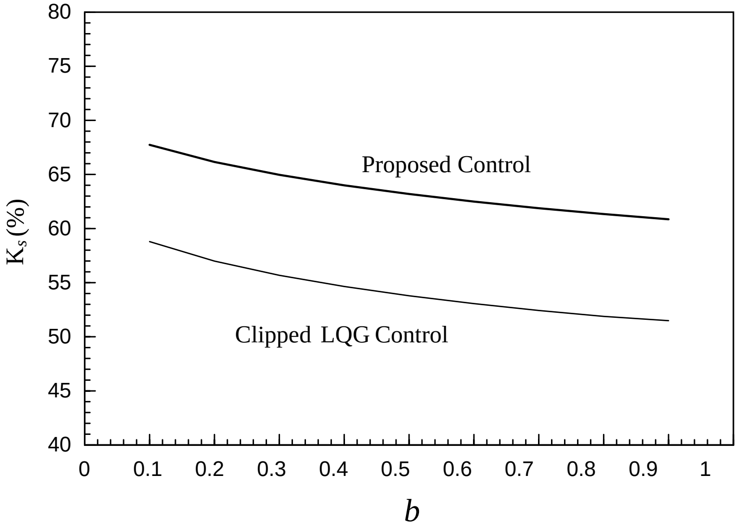

Figure 1. Percentage reduction in root-mean-square displacement versus non-linear stiffness for system (42) ($D = 0{\cdot}3$).

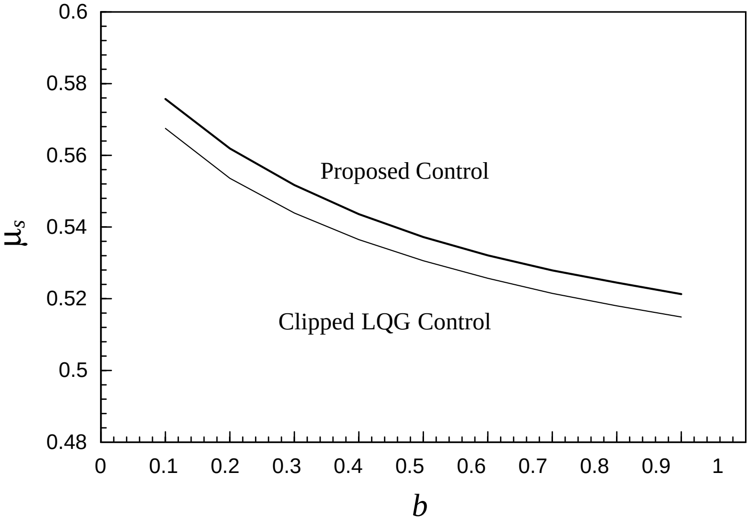

Figure 2. Control efficiency versus non-linear stiffness for system (42) ($D = 0{\cdot}3$).

and $\eta(t)$ are horizontal and vertical random ground acceleration excitations that act as external and parametric excitations and, for simplicity, are idealized as Gaussian white noises with intensities $2D_1$ and $2D_2$ respectively. Z represents the hysteretic component of the restoring force and is herein modelled by a non-linear differential equation [25, 26]:

$$\dot{Z} = A\dot{X} - \beta\dot{X}|Z|^n - \gamma|\dot{X}|Z|Z|^{n-1}, \tag{75}$$

where A, β, γ and n are the hysteresis parameters. The optimal non-linear stochastic control of hysteretic column has been studied [11]. Here, the control force u is produced by

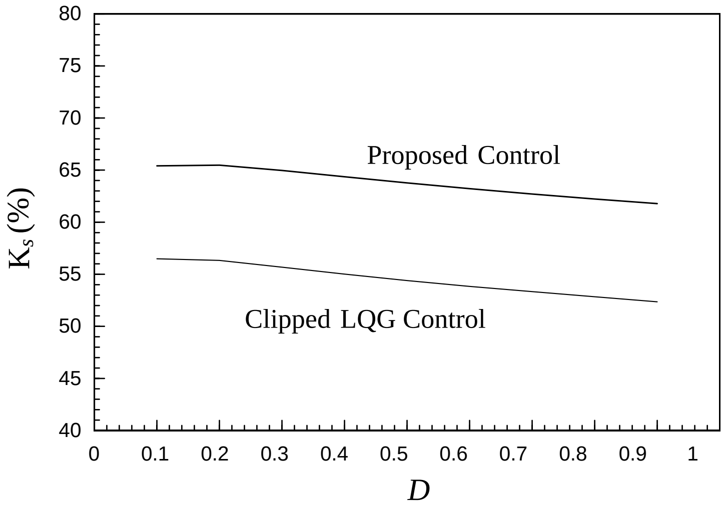

Figure 3. Percentage reduction in root-mean-square displacement versus excitation intensity for system (42) ($b = 0{\cdot}3$).

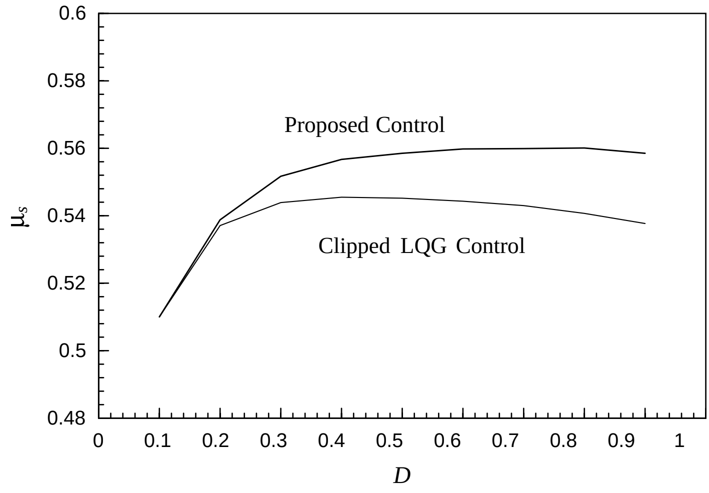

Figure 4. Control efficiency versus excitation intensity for system (42) ($b = 0{\cdot}3$).

an ER or MR damper and the stochastic optimal semi-active control of the hysteretic column is studied.

Combining the passive control force $-c_1\dot{X}$ of the ER or MR damper with the viscous damping force $-2\zeta_0\dot{X}$ of the system, equation (74) can be rewritten as

$$\ddot{X} + 2\zeta\dot{X} + [\alpha - k_1 - k_2\eta(t)]X + (1-\alpha)Z = \xi(t) + u_a. \tag{76}$$

By using the stochastic averaging method [24], the partially completed averaged Itô equation is obtained as follows:

$$\mathrm{d}H = \left[m(H) + \left\langle \frac{\partial H}{\partial P} u \right\rangle \right] \mathrm{d}t + \sigma(H)\,\mathrm{dB}(t), \tag{77}$$

where

$$H = \dot{x}^2/2 + U(x),$$

$$m(H) = \frac{1}{T(H)}\left[-A_r - 4\zeta\int_{-a}^{a}\sqrt{2H-2U(x)}\,\mathrm{d}x + 2k_2^2 D_2\int_{-a}^{a}\frac{x^2\,\mathrm{d}x}{\sqrt{2H-2U(x)}}\right] + D_1,$$

$$\sigma^2(H) = \frac{2}{T(H)}\int_{-a}^{a}(2D_1 + 2k_2^2 D_2 x^2)\sqrt{2H-2U(x)}\,\mathrm{d}x,$$

$$T(H) = 2\int_{-a}^{a}\frac{\mathrm{d}x}{\sqrt{2H-2U(x)}}. \tag{78}$$

U is the equivalent potential energy of the system, A_r is the area of hysteresis loop and a is the amplitude of displacement and related to H by $H = U(\pm a)$. They all depend on hysteresis parameters. In the case $n = A = 1$, the expressions for $U(x)$, A_r and a are given in Appendix A.

Assume that the cost function is of the form of that in equations (47) and (48). Then, the optimal control force u_a^* is still of the form of that in equations (49) and (50). For ergodic control, $\mathrm{d}V/\mathrm{d}H$ is also governed by the final dynamical programming equation (51) but with

$$G(H) = \frac{2}{T(H)}\int_{-a}^{a}\sqrt{2H-2U(x)}\,\mathrm{d}x, \tag{79}$$

where $T(H)$ is given in equation (78). Set the parameters s_0, s_1, s_2, s_3 in equation (48) and R in equation (47) so that $\mathrm{d}V/\mathrm{d}H \geqslant 0$. Then the ER or MR damper will produce the optimal control force u_a^* all the time.

Solving equation (51) and substituting the resultant $\mathrm{d}V/\mathrm{d}H$ into equation (49) yield u_a^*. Then, substituting u_a^* into equation (77) to replace u_a and averaging $(\partial H/\partial P)u_a^*$ lead to the fully averaged Itô equation for H. The stationary probability density of H and the mean square displacement of the semi-actively controlled hysteretic column, and the mean square active control force component can be obtained by using equations (52), (54) and (56) respectively. Some numerical result is obtained for $\zeta = 0{\cdot}025$, $k_1 = 0{\cdot}04$, $k_2 = 0{\cdot}1$, $\alpha = \beta = \gamma = 0{\cdot}5$, $A = n = 1$, $D_2 = 0{\cdot}1$, $R = 1$, $s_1 = s_3 = 0$, $s_2 = 1{\cdot}0$, $\mathrm{d}V/\mathrm{d}H|_{H=0} = 3{\cdot}5$ and is shown in Figure 5. It is seen from the comparison of this figure with Figures 3 and 4 that the effectiveness and efficiency of the proposed stochastic optimal semi-active control strategy can be even better for this more complicated system.

8. CONCLUSIONS

In the present paper, a stochastic optimal semi-active control strategy for ER/MR dampers has been proposed. The strategy is developed from the non-linear stochastic optimal control strategy previously proposed by present authors based on the stochastic averaging method for quasi-Hamiltonian systems and the stochastic dynamical programming principle. The major advantage of the proposed stochastic optimal semi-active control strategy is that it is possible to implement the non-linear stochastic optimal active control strategy using ER/MR dampers without clipping even for non-linear stochastic systems with m.d.o.f. In this case, the responses of semi-actively and passively controlled systems can be predicted analytically and compared with each other. However, using the semi-active control strategy based on LQG, the desired optimal control forces cannot always be produced by ER/MR dampers and clipping is usually necessary, especially for linear stochastic systems with m.d.o.f. Therefore, the effectiveness and efficiency of the

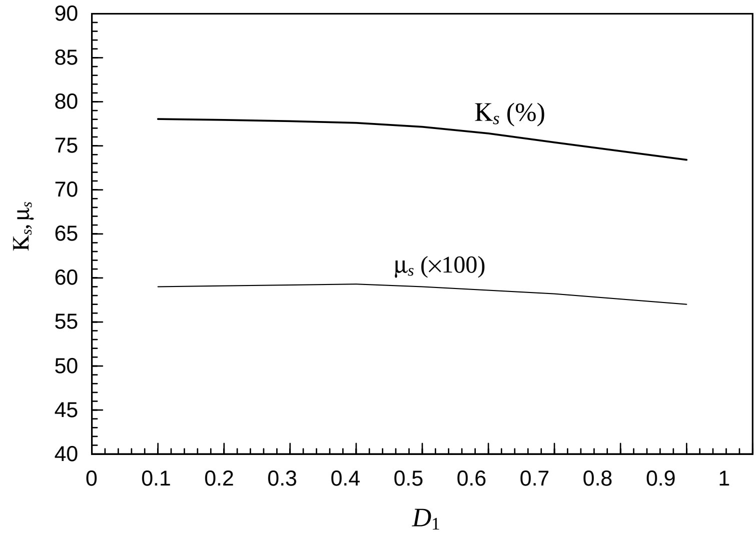

Figure 5. Effectiveness K_s and efficiency μ_s of the proposed semi-active control strategy versus excitation intensity for system (74).

proposed stochastic optimal semi-active control strategy are usually higher than those of the clipped LQG control strategy. This has been illustrated with the numerical results for randomly excited Duffing oscillator.

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REFERENCES

1. M. D. Symans and M. C. Constantinon 1999 *Engineering Structures* **21**, 469–487. Semi-active control systems for seismic protection of structures: a state-of-the-art review.
2. L. M. Jansen and S. J. Dyke 2000 *American Society of Civil Engineers Journal of Engineering Mechanics* **126**, 795–803. Semiactive control strategies for MR dampers: comparative study.
3. W. L. Brogan 1991 *Modern Control Theory*. Englewood Cliffs, NJ: Prentice-Hall.
4. G. Leitmann 1994 *Journal of Intelligent Material Systems and Structures* **5**, 841–846. Semiactive control for vibration attenuation.
5. N. H. McClamroch and H. P. Gavin 1995 *Proceedings of the American Control Conference*, 4173–4177. Washington: American Automatic Control Council. Closed loop structural control using electrorheological dampers.
6. J. A. Inaudi 1997 *Earthquake Engineering and Structural Dynamics* **26**, 361–376. Modulated homogeneous friction: a semi-active damping strategy.
7. R. L. Sack and W. Patten 1994 *Proceedings of the International Workshop on Structural Control*, 417–431. Los Angeles: University of Southern California. Semiactive hydraulic structural control.
8. S. J. Dyke, B. F. Spencer Jr, M. K. Sain and J. D. Carlson 1996 *Smart Materials and Structures* **5**, 565–575. Modeling and control of magnetorheological dampers for seismic response reduction.
9. W. Q. Zhu and Z. G. Ying 1999 *Science in China Series A* **42**, 1213–1219. Optimal nonlinear feedback control of quasi-Hamiltonian systems.

10. W. Q. ZHU, Z. G. YING and T. T. SOONG 1999 in *Stochastic Structural Dynamics* (B. F. Spencer Jr and E.A. Johnson, editors), 141–148. Rotterdam: Balkema. Optimal nonlinear feedback control of structures under random loading.
11. W. Q. ZHU, Z. G. YING, Y. Q. NI and J. M. KO 2000 *Journal of Engineering Mechanics American Society of Civil Engineers* **126**, 1027–1032. Optimal nonlinear stochastic control of hysteretic systems.
12. W. Q. ZHU, Z. G. YING and T. T. SOONG 2001 *Nonlinear Dynamics* **24**, 31–51. An optimal nonlinear feedback control strategy for randomly excited structural systems.
13. W. Q. ZHU and Y. Q. YANG 1997 *American Society of Mechanical Engineers Journal of Applied Mechanics* **64**, 157–164. Stochastic averaging of quasi-nonintegrable-Hamiltonian systems.
14. W. Q. ZHU, Z. L. HUANG and Y. Q. YANG 1997 *American Society of Mechanical Engineers Journal of Applied Mechanics* **64**, 975–984. Stochastic averaging of quasi-integrable-Hamiltonian systems.
15. W. Q. ZHU, Z. L. HUANG and Y. SUZUKI 2002 *International Journal of Non-Linear Mechanics* **31**, 419–437. Stochastic averaging and Lyapunov exponent of quasi partially integrable Hamiltonian systems.
16. W. H. FLEMING and R. W. RISHEL 1975 *Deterministic and Stochastic Optimal Control*. Berlin: Springer-Verlag.
17. W. H. FLEMING and H. M. SONER 1992 *Controlled Markov Process and Viscosity Solutions*. New York: Springer-Verlag.
18. J. M. YONG and X. Y. ZHOU 1999 *Stochastic Controls, Hamiltonian Systems and HJB Equations*. New York: Springer-Verlag.
19. W. Q. ZHU and Z. G. YING 2002 *Engineering Structures* **24**, 333–342. Nonlinear stochastic optimal control of partially observable linear structures.
20. W. Q. ZHU, Z. L. HUANG and M. L. DENG 2002 *International Journal of Non-Linear Mechanics* **37**, 1057–1071. Feedback minimization of first-passage failure of quasi non-integrable Hamiltonian systems.
21. H. P. GAVIN, R. D. HANSON and F. E. FILISKO 1996 *American Society of Mechanical Engineers Journal of Applied Mechanics* **63**, 676–682. Electrorheological dampers. Part II: testing and modeling.
22. B. F. SPENCER, S. J. DYKE, M. K. SAIN and J. D. CARLSON 1997 *American Society of Civil Engineers Journal of Engineering Mechanics* **123**, 230–238. Phenomenological model for magneto-rheological dampers.
23. N. M. WERELEY, L. PANG and G. M. KAMATH 1998 *Journal of Intelligent Material Systems and Structures* **9**, 642–649. Idealized hysteresis modeling of electro-rheological and magnetorheological dampers.
24. W. Q. ZHU and Y. K. LIN 1991 *American Society of Civil Engineers Journal of Engineering Mechanics* **117**, 1890–1905. Stochastic averaging of energy envelope.
25. R. BOUC 1967 *Proceedings of 4th Conference on Non-Linear Oscillation Prague, Czechoslovakia* (abstract), 315–315. Forced vibration of mechanical system with hysteresis.
26. Y. K. WEN 1976 *American Society of Civil Engineers Journal of Engineering Mechanics Division* **102**, 249–263. Method for random vibration of hysteretic systems.

APPENDIX A: EXPRESSIONS FOR $U(X)$, A_R AND a

In the case $n = A = 1$, the expressions for $U(x)$ and A_r are

$$U(x) = \begin{cases} \frac{1}{2}(\alpha - k_1)x^2 + \frac{1-\alpha}{\gamma-\beta}\Big\{x + x_0 \\ \quad + \frac{1}{\gamma-\beta}[\mathrm{e}^{-(\gamma-\beta)(x+x_0)} - 1]\Big\}, & -a \leqslant x \leqslant -x_0, \gamma \neq \pm\beta, \\ \frac{1}{2}(\alpha - k_1)x^2 + \frac{1-\alpha}{\gamma^2-\beta^2}\Big\{1 - \mathrm{e}^{-(\gamma-\beta)(x+x_0)} \\ \quad - \frac{\gamma+\beta}{\gamma-\beta}\ln\left[1 + \frac{\gamma+\beta}{\gamma-\beta}(1 - \mathrm{e}^{-(\gamma-\beta)(x+x_0)})\right]\Big\}, & -x_0 \leqslant x \leqslant a, \gamma \neq \pm\beta, \end{cases} \tag{A1}$$

$$U(x) = \begin{cases} \frac{1}{2}(\alpha - k_1)x^2 + \frac{1}{2}(1-\alpha)(x+x_0)^2, & -a \leqslant x \leqslant -x_0, \gamma = \beta, \\ \frac{1}{2}(\alpha - k_1)x^2 + \frac{1}{8\gamma^2}(1-\alpha)[1 - \mathrm{e}^{-2\gamma(x+x_0)}]^2, & -x_0 \leqslant x \leqslant a, \gamma = \beta, \end{cases} \tag{A2}$$

$$U(x) = \begin{cases} \dfrac{1}{2}(\alpha - k_1)x^2 + \dfrac{1-\alpha}{2\gamma}(x+x_0) + \dfrac{1-\alpha}{4\gamma^2}[\mathrm{e}^{-2\gamma(x+x_0)} - 1], & -a \leqslant x \leqslant -x_0, \gamma = -\beta, \\ \dfrac{1}{2}(\alpha - k_1)x^2 + \dfrac{1-\alpha}{2\gamma}\left[x + x_0 - \dfrac{1}{2\gamma}\ln[1 + 2\gamma(x+x_0)]\right], & -x_0 \leqslant x \leqslant a, \gamma = -\beta, \end{cases} \tag{A3}$$

$$A_r = \frac{4}{\gamma^2 - \beta^2}(1-\alpha)\left\{\gamma a - \beta x_0 + \frac{\gamma}{\gamma+\beta}[\mathrm{e}^{-(\gamma+\beta)(a+x_0)} - 1]\right\}, \quad \gamma \neq \pm\beta, \tag{A4}$$

$$A_r = (1-\alpha)[2x_0/\gamma - (a-x_0)^2], \quad \gamma = \beta, \tag{A5}$$

$$A_r = (1-\alpha)[-2x_0/\gamma + (a-x_0)^2], \quad \gamma = -\beta, \tag{A6}$$

where x_0 is the residual hysteresis displacement. The quantities a and x_0 can be obtained for given H by solving the following equations:

$$(\gamma+\beta)\mathrm{e}^{(\gamma-\beta)(a-x_0)} + (\gamma-\beta)\mathrm{e}^{-(\gamma+\beta)(a+x_0)} = 2\gamma, \quad \gamma \neq \pm\beta, \tag{A7}$$

$$2\gamma(a-x_0) = 1 - \mathrm{e}^{-2\gamma(a+x_0)}, \quad \gamma = \beta, \tag{A8}$$

$$2\gamma(a+x_0) = -1 + \mathrm{e}^{2\gamma(a-x_0)}, \quad \gamma = -\beta \tag{A9}$$

and

$$2H - (\alpha - k_1)a^2 = (1-\alpha)[-a + x_0 + (\mathrm{e}^{-(\gamma-\beta)(a-x_0)} - 1)/(\gamma-\beta)]/(\gamma-\beta), \quad \gamma \neq \pm\beta, \tag{A10}$$

$$2H - (\alpha - k_1)a^2 = (1-\alpha)(a-x_0)^2, \quad \gamma = \beta, \tag{A11}$$

$$2H - (\alpha - k_1)a^2 = -(1-\alpha)(a-x_0)/2\gamma + (1-\alpha)[\mathrm{e}^{2\gamma(a-x_0)} - 1]/4\gamma^2, \quad \gamma = -\beta. \tag{A12}$$

APPENDIX B: NOMENCLATURE

A	displacement amplitude or hysteresis parameter
$\mathbf{A}$	coefficient matrix of state equation
A_r	area of hysteresis loop
a	linear stiffness or displacement amplitude
b	non-linear stiffness
b_{ir}	damper placement coefficients
C_p	normalization constant
C_s	normalization constant
C_{ra}	positive constants of ER/MR models
c_{ij}	damping coefficients
$\mathbf{D}$	intensity matrix of random excitations
D_{kl}	intensities of Gaussian white noises
E	expectation
F_r	absolute values of active control force components
F_{ik}	excitation magnitudes
G	function of system energy
g	energy function in cost function
H	Hamiltonian or total system energy
$\mathbf{H}$	Hamiltonian vector
J	performance index

L	cost function
m	number of random excitations
$m(H)$	drift coefficient
n	degree of freedom of system or hysteresis parameter
$\mathbf{P}$	generalized momentum vector or constant matrix in value function
p_p	probability density of passively controlled system
p_s	probability density of semi-actively controlled system
$\mathbf{Q}$	generalized displacement vector
$\mathbf{R}$	weight matrix in cost function
$\mathbf{S}$	weight matrix in cost function
s	number of ER/MR dampers
s_0	weight in cost function
s_1	weight in cost function
s_2	weight in cost function
s_3	weight in cost function
T	quasi-period
t_f	final time
U	equivalent potential energy
u_r	control forces of ER/MR dampers
u_{ra}	active control force components of ER/MR dampers due to external power
u_{rp}	passive control force components of ER/MR dampers without external power
u_{ra}^*	optimal control forces
$\mathbf{u}_a$	control force vector
V	value function
V_{re}	external voltages
$\mathbf{W}$	covariance matrix
X	displacement
$\mathbf{Y}$	state vector
Z	hysteretic force
α	stiffness ratio
α_r	positive constants of ER/MR models
β	hysteresis parameter
$\mathrm{B}_k(t)$	unit Wiener processes
γ	hysteresis parameter
$\eta(t)$	vertical ground acceleration excitation
K_s	percentage reduction of root-mean-square responses
λ	optimal average cost
μ_s	control efficiency
$\sigma(H)$	diffusion coefficient
$\xi(t)$	Gaussian white noise or horizontal ground acceleration excitation
$\xi_k(t)$	random excitations
Ψ	final cost
ζ	viscous damping ratio.